Math 150, *Lecture Notes- Bonds* **Name_____________**

Section 2.4 *The Chain Rule*

THEOREM 2.10 The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

PROOF Let $h(x) = f(g(x))$. Then, using the alternative form of the derivative, you need to show that, for $x = c$,

$$h'(c) = f'(g(c))g'(c).$$

An important consideration in this proof is the behavior of g as x approaches c. A problem occurs if there are values of x, other than c, such that $g(x) = g(c)$. Appendix A shows how to use the differentiability of f and g to overcome this problem. For now, assume that $g(x) \neq g(c)$ for values of x other than c. In the proofs of the Product Rule and the Quotient Rule, the same quantity was added and subtracted to obtain the desired form. This proof uses a similar technique—multiplying and dividing by the same (nonzero) quantity. Note that because g is differentiable, it is also continuous, and it follows that $g(x) \to g(c)$ as $x \to c$.

$$\begin{aligned}
h'(c) &= \lim_{x\to c} \frac{f(g(x)) - f(g(c))}{x - c} \\
&= \lim_{x\to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c}\right], \quad g(x) \neq g(c) \\
&= \left[\lim_{x\to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)}\right]\left[\lim_{x\to c} \frac{g(x) - g(c)}{x - c}\right] \\
&= f'(g(c))g'(c)
\end{aligned}$$

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When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts—an inner part and an outer part.

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

The derivative of $y = f(u)$ is the derivative of the outer function (at the inner function u) *times* the derivative of the inner function.

$$y' = f'(u) \cdot u'$$

Ex.1 Writing the decomposition of a composite function.

	$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a.	$y = \dfrac{1}{x+1}$	$u = x + 1$	$y = \dfrac{1}{u}$
b.	$y = \sin 2x$	$u = 2x$	$y = \sin u$
c.	$y = \sqrt{3x^2 - x + 1}$	$u = 3x^2 - x + 1$	$y = \sqrt{u}$
d.	$y = \tan^2 x$	$u = \tan x$	$y = u^2$

Ex.2 Find the derivative of $y = 5\left(2 - x^3\right)^4$.

$$\frac{dy}{dx} = 5 \cdot \left\{4(2-x^3)^3\right\} \frac{d}{dx}(2-x^3)$$

$$\frac{dy}{dx} = 20(2-x^3)^3 \cdot (-3x^2)$$

$$\frac{dy}{dx} = -60x^2(2-x^3)^3$$

THEOREM 2.11 The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = n u^{n-1} u'.$$

PROOF Because $y = u^n$, you apply the Chain Rule to obtain

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right) = \frac{d}{du}[u^n]\frac{du}{dx}.$$

By the (Simple) Power Rule in Section 2.2, you have $D_u[u^n] = n u^{n-1}$, and it follows that

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}.$$

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Ex.3 Find the derivative of $g(t)=8\sqrt[4]{9-t^2}$.

$$g(t) = 8(9-t^2)^{1/4}$$

$$g'(t) = 8\cdot\left[\tfrac{1}{4}(9-t^2)^{-3/4}\right]\cdot\frac{d}{dt}(9-t^2)$$

$$g'(t) = 2(9-t^2)^{-3/4}\cdot(-2t) \longleftarrow$$

$$g'(t) = \frac{-4t}{(9-t^2)^{3/4}}$$

Summary of Differentiation Rules

General Differentiation Rules	Let f, g, and u be differentiable functions of x.		
	Constant Multiple Rule: $\frac{d}{dx}[cf] = cf'$	*Sum or Difference Rule*: $\frac{d}{dx}[f \pm g] = f' \pm g'$	
	Product Rule: $\frac{d}{dx}[fg] = fg' + gf'$	*Quotient Rule*: $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$	
Derivatives of Algebraic Functions	*Constant Rule*: $\frac{d}{dx}[c] = 0$	*(Simple) Power Rule*: $\frac{d}{dx}[x^n] = nx^{n-1}$, $\frac{d}{dx}[x] = 1$	
Derivatives of Trigonometric Functions	$\frac{d}{dx}[\sin x] = \cos x$	$\frac{d}{dx}[\tan x] = \sec^2 x$	$\frac{d}{dx}[\sec x] = \sec x \tan x$
	$\frac{d}{dx}[\cos x] = -\sin x$	$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\frac{d}{dx}[\csc x] = -\csc x \cot x$
Chain Rule	*Chain Rule*: $\frac{d}{dx}[f(u)] = f'(u)\,u'$	*General Power Rule*: $\frac{d}{dx}[u^n] = nu^{n-1}u'$	

Ex.4 Find the derivative of $y=x^2\sqrt{16-x^2}$.

$$\frac{d}{dx}[y] = \frac{d}{dx}\left[x^2\cdot(16-x^2)^{1/2}\right]$$

$$\frac{dy}{dx} = x^2\frac{d}{dx}\left[(16-x^2)^{1/2}\right] + (16-x^2)^{1/2}\frac{d}{dx}\left[x^2\right]$$

$$\frac{dy}{dx} = x^2\cdot\left[\tfrac{1}{2}(16-x^2)^{-1/2}\right]\cdot\frac{d}{dx}\left[16-x^2\right] + (16-x^2)^{1/2}\cdot\left[2x\right]$$

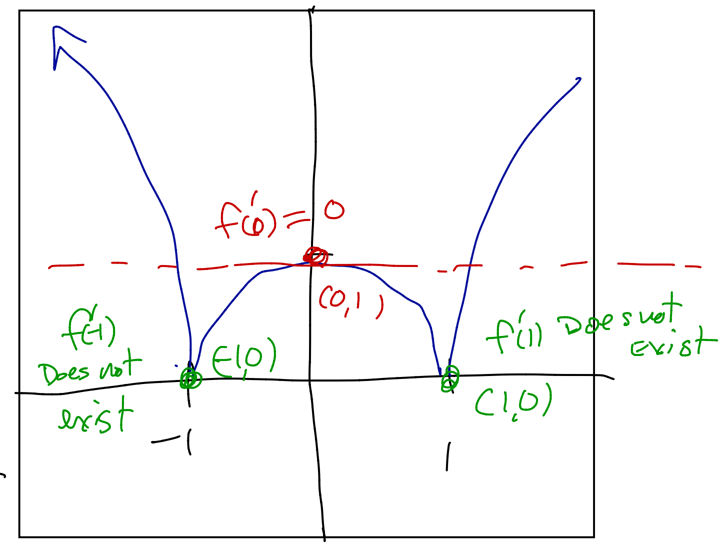
$$\frac{dy}{dx} = x^2\cdot\left[\tfrac{1}{2}(16-x^2)^{-1/2}\right]\cdot\left[-2x\right] + (16-x^2)^{1/2}\cdot\left[2x\right]$$

$$\frac{dy}{dx} = \left[x(16-x^2)^{-1/2}\right]\left[-x^2+2(16-x^2)\right]$$

$$\frac{dy}{dx} = \frac{x\left[-x^2+32-2x^2\right]}{(16-x^2)^{1/2}}$$

$$\frac{dy}{dx} = \frac{x\left[-3x^2+32\right]}{(16-x^2)^{1/2}}$$

Ex.5 Find on the graph of $f(x)=\sqrt[3]{\left(x^2-1\right)^2}$ for which $f'(x)=0$ and those for which $f'(x)$ does not exist.



$f(x)=(x^2-1)^{2/3}$

$\frac{d}{dx}[f(x)]=\frac{d}{dx}[(x^2-1)^{2/3}]$

$f'(x)=\frac{2}{3}\cdot(x^2-1)^{-1/3}\cdot\frac{d}{dx}(x^2-1)$

$f'(x)=\frac{2}{3}\cdot(x^2-1)^{-1/3}\cdot[2x]$

$f'(x)=\frac{4x}{3\cdot(x^2-1)^{1/3}}$

$f'(x)=0$

$0=\frac{4x}{3(x^2-1)^{1/3}}$

$0=4x$

$x=0$

$f'(x)$ does not exist

$3(x^2-1)^{1/3}=0$

$(x^2-1)^{1/3}=0$

$x^2-1=0$

$(x-1)(x+1)=0$

$x=1$ or $x=-1$

Ex.6 Find the derivative of $y=\frac{t}{\sqrt{t^4+4}}$.

$y=t(t^4+4)^{-1/2}$

$\frac{dy}{dt}=\frac{d}{dt}\left[t\cdot(t^4+4)^{-\frac{1}{2}}\right]$

$\frac{dy}{dt}=(t)\cdot\frac{d}{dt}\left[(t^4+4)^{-\frac{1}{2}}\right]+(t^4+4)^{-1/2}\cdot\frac{d}{dt}(t)$

$\frac{dy}{dt}=t\cdot\left[-\frac{1}{2}(t^4+4)^{-3/2}\cdot 4t^3\right]+(t^4+4)^{-1/2}\cdot 1$

$\frac{dy}{dt}=(t^4+4)^{-3/2}\left[-2t^4+(t^4+4)\right]$

$\boxed{\frac{dy}{dt}=\frac{-t^4+4}{(t^4+4)^{3/2}}}$

Ex.7 Find the derivative of $h(t)=\left(\dfrac{t^2}{t^3+2}\right)^2$.

$$\frac{d}{dt}\left[h(t)\right]=\frac{d}{dt}\left[\left(\frac{t^2}{t^3+2}\right)^2\right]$$

$$=2\left(\frac{t^2}{t^3+2}\right)^1\cdot\frac{d}{dt}\left[\frac{t^2}{t^3+2}\right]$$

$$=\frac{2t^2}{t^3+2}\cdot\frac{(t^3+2)\frac{d}{dt}(t^2)-(t^2)\frac{d}{dt}(t^3+2)}{(t^3+2)^2}$$

$$=\frac{2t^2\left[(t^3+2)(2t)-(t^2)(3t^2)\right]}{(t^3+2)^3}$$

$$=\frac{2t^2\left[2t^4+4t-3t^4\right]}{(t^3+2)^3}$$

$$=\frac{2t^2(-t^4+4t)}{(t^3+2)^3}$$

$$=\frac{2t^3(-t^3+4)}{(t^3+2)^3}$$

Trigonometric Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions are as follows.

$$\frac{d}{dx}[\sin u]=(\cos u)\,u' \qquad \frac{d}{dx}[\cos u]=-(\sin u)\,u'$$

$$\frac{d}{dx}[\tan u]=(\sec^2 u)\,u' \qquad \frac{d}{dx}[\cot u]=-(\csc^2 u)\,u'$$

$$\frac{d}{dx}[\sec u]=(\sec u\tan u)\,u' \qquad \frac{d}{dx}[\csc u]=-(\csc u\cot u)\,u'$$

Ex.8 Applying the Chain Rule to Trigonometric Functions

a. $y=\sin \overbrace{2x}^{u}$ $\qquad y'=\overbrace{\cos 2x}^{\cos u}\,\overbrace{\frac{d}{dx}[2x]}^{u'}=(\cos 2x)(2)=2\cos 2x$

b. $y=\cos(x-1)$ $\qquad y'=-\sin(x-1)$

c. $y=\tan 3x$ $\qquad y'=3\sec^2 3x$

Ex.9 Derivatives, Parentheses, and Trigonometric Functions

Find the derivative of the following functions:

(a) $y = \cos 3x^2 = \cos(3x^2)$

(b) $y = (\cos 3)x^2$

(c) $y = \cos(3x)^2 = \cos(9x^2)$

(d) $y = \cos^2 x = (\cos x)^2$

Test Question??

Ex.10 Find the derivative of $g(\theta) = 5\cos^2(\pi\theta)$.

$$\frac{d}{d\theta}\{g(\theta)\} = 5 \cdot \frac{d}{d\theta}[\cos^2(\pi\theta)]$$

$$g'(\theta) = 5 \cdot [2 \cdot \cos^1(\pi\theta)] \cdot \frac{d}{d\theta}[\cos(\pi\theta)]$$

$$g'(\theta) = 10\cos(\pi\theta) \cdot [-\sin(\pi\theta)] \cdot \frac{d}{d\theta}[\pi\theta]$$

$$g'(\theta) = -10\cos(\pi\theta)\sin(\pi\theta) \cdot \pi$$

$$g'(\theta) = -10\pi\cos(\pi\theta)\sin(\pi\theta)$$

Test Question?

Ex.11 Find the derivative of $g(\theta)=\cos\sqrt{\sin(\tan(\pi\theta))}$.

$$g'(\theta) = -\sin\sqrt{\sin(\tan(\pi\theta))}\cdot\frac{d}{d\theta}\left(\sin(\tan(\pi\theta))\right)^{1/2}$$

$$g'(\theta) = -\sin\sqrt{\sin(\tan(\pi\theta))}\cdot\frac{1}{2}\left(\sin(\tan(\pi\theta))\right)^{-1/2}\cdot\frac{d}{d\theta}\left[\sin(\tan(\pi\theta))\right]$$

$$g'(\theta) = -\frac{1}{2}\sin\sqrt{\sin(\tan(\pi\theta))}\cdot\frac{1}{\sqrt{\sin(\tan(\pi\theta))}}\cdot\cos(\tan(\pi\theta))\cdot\frac{d}{d\theta}\left[\tan(\pi\theta)\right]$$

$$g'(\theta) = \frac{-\sin\sqrt{\sin(\tan(\pi\theta))}\cdot\cos\tan(\pi\theta)\cdot\sec^2(\pi\theta)\cdot\pi}{\sqrt{\sin(\tan(\pi\theta))}}$$

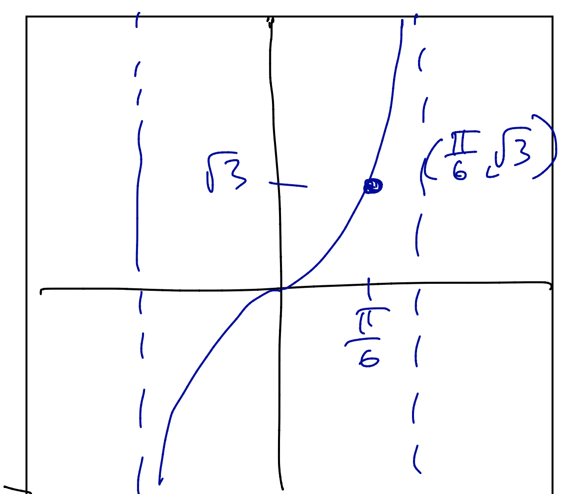
Ex.12 Evaluate the second derivative of $g(\theta)=\tan(2\theta)$ at $\left(\frac{\pi}{6},\sqrt{3}\right)$.

$$\frac{d}{d\theta}[g(\theta)] = \frac{d}{d\theta}[\tan(2\theta)]$$

$$g'(\theta) = \sec^2(2\theta)\cdot\frac{d}{d\theta}[2\theta]$$

$$g'(\theta) = \sec^2(2\theta)\cdot 2$$

$$g'(\theta) = 2\sec^2(2\theta)$$



$$\frac{d}{d\theta}[g'(\theta)] = \frac{d}{d\theta}[2\sec^2(2\theta)]$$

$$g''(\theta) = 2\cdot[2\sec^1(2\theta)]\cdot\frac{d}{d\theta}[\sec(2\theta)]$$

$$g''(\theta) = 4\sec(2\theta)\cdot\sec(2\theta)\cdot\tan(2\theta)\cdot\frac{d}{d\theta}(2\theta)$$

$$g''(\theta) = 4\sec^2(2\theta)\tan(2\theta)\cdot 2$$

$$g''(\theta) = 8\sec^2(2\theta)\tan(2\theta)$$

$$g''\left(\frac{\pi}{6}\right) = 8\left[\frac{1}{\cos^2(2\cdot\frac{\pi}{6})}\right]\left[\frac{\sin(2\cdot\frac{\pi}{6})}{\cos(2\cdot\frac{\pi}{6})}\right]$$

$$g''\left(\frac{\pi}{6}\right) = 8\cdot\left[\frac{\sin(\pi/3)}{\cos^3(\frac{\pi}{3})}\right]$$

$$g''\left(\frac{\pi}{6}\right) = 8\left[\frac{\frac{\sqrt{3}}{2}}{\left(\frac{1}{2}\right)^3}\right]$$

$$g''\left(\frac{\pi}{6}\right) = \frac{4\sqrt{3}}{\frac{1}{8}} = \frac{4\sqrt{3}}{1}\cdot\frac{8}{1}$$

$$g''\left(\frac{\pi}{6}\right) = 32\sqrt{3}$$

Test Question??

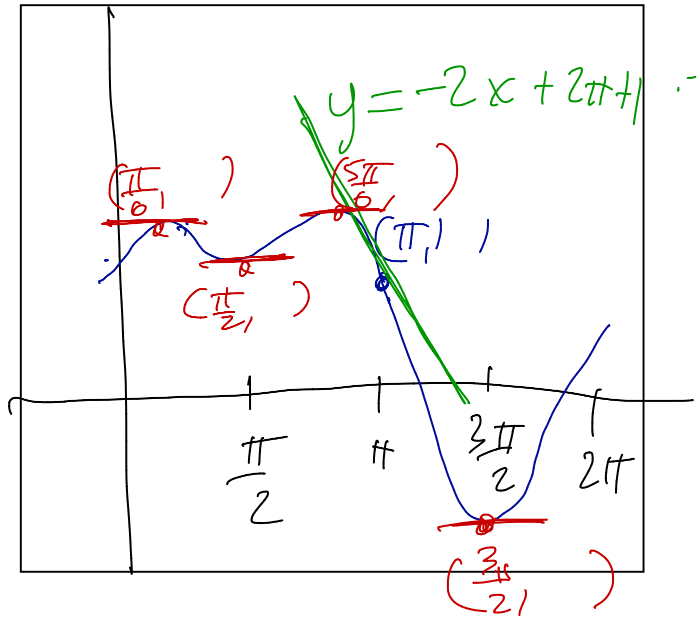
$y - y_1 = m_{tan}(x - x_1)$

(x_1, y_1)

Ex.13 Find the equation of the tangent line to the graph of $f(x) = 2\sin(x) + \cos(2x)$ at $(\pi, 1)$. Then, find all values of x in $(0, 2\pi)$ at which the graph of f has a horizontal tangent.

$f'(x) = 2 \cdot \cos(x) + [-\sin(2x)] \cdot 2$ Chain Rule

$\boxed{f'(x) = 2\cos(x) - 2\sin(2x)}$



$m_{tan}\Big|_{(\pi,1)} = f'(\pi)$

$= 2\cos(\pi) - 2\sin(2\pi)$

$= 2(-1) - 2 \cdot 0$

$= -2$

$y - y_1 = m_{tan}(x - x_1)$

$y - (1) = -2[x - (\pi)]$

$y - 1 = -2x + 2\pi$

$\boxed{y = -2x + 2\pi + 1}$

$m_{tan} = 0$ at a Horizontal Tangent line

$f'(x) = 0$

Solve: $2\cos(x) - 2\sin(2x) = 0$

$2\cos(x) - 2 \cdot 2\sin(x)\cos(x) = 0$

$2\cos(x)[1 - 2\sin(x)] = 0$

Either

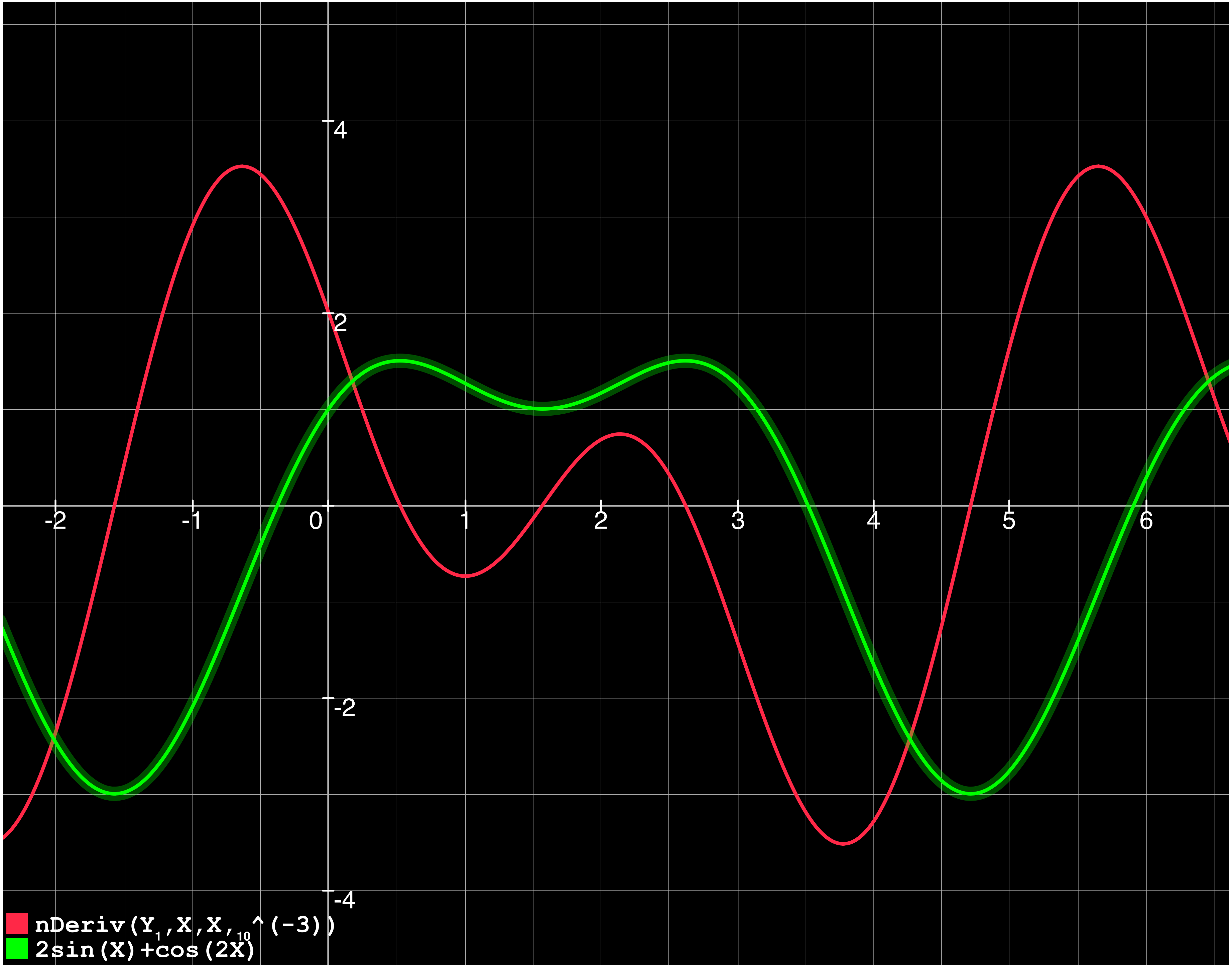
$2\cos(x) = 0$, or $1 - 2\sin(x) = 0$

$\cos(x) = 0$ | $1 = 2\sin(x)$

$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$ | $\frac{1}{2} = \sin(x)$

$0 < x < 2\pi$ | $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

$0 < x < 2\pi$


4
2
-2
-4
-2
-1
0
1
2
3
4
5
6
nDeriv(Y1,X,X,10^(-3))
2sin(X)+cos(2X)

Ex.14 Given $h(x)=f(g(x))$ and $s(x)=g(f(x))$, use the graphs of f and g to find the following derivatives:

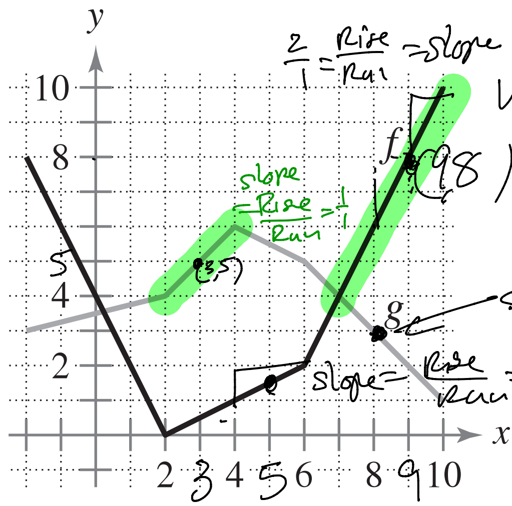
$h'(x) = f'(g(x)) \cdot g'(x)$ | $s'(x) = g'(f(x)) \cdot f'(x)$

(a) Find $h'(3)$.

(b) Find $s'(9)$.

$h'(3) = f'(g(3)) \cdot g'(3)$

$h'(3) = f'(5) \cdot 1$



$h'(3) = \frac{1}{2} \cdot 1$

$\boxed{h'(3) = \frac{1}{2}}$

$s'(9) = g'(f(9)) \cdot f'(9)$

$= g'(8) \cdot f'(9)$

$= -1 \cdot \left(\frac{2}{1}\right)$

$s'(9) = -2$

#58 $h(t) = 2\cot^2(\pi t + 2)$

$$\frac{d}{dt}[h(t)] = \frac{d}{dt}[2\cot^2(\pi t + 2)]$$

$$h'(t) = 2 \cdot \frac{d}{dt}[\cot^2(\pi t + 2)]$$

$$h'(t) = 2 \cdot [2 \cdot \cot^1(\pi t + 2)] \cdot \frac{d}{dt}[\cot(\pi t + 2)]$$

$$h'(t) = 2 \cdot [2 \cdot \cot^1(\pi t + 2)] \cdot [-\csc^2(\pi t + 2)] \cdot \frac{d}{dt}(\pi t + 2)$$

$$h'(t) = 2 \cdot [2 \cdot \cot^1(\pi t + 2)] \cdot [-\csc^2(\pi t + 2)] \cdot \pi$$

$$h'(t) = -4\pi \cot(\pi t + 2) \csc^2(\pi t + 2)$$

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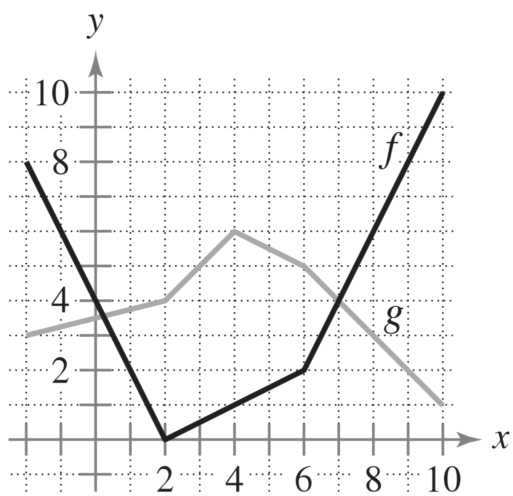

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